^{1} Contrôle continu N^0 : 2

Exercice 1: Résoudre les équations suivantes :

a) $2y'' - 7y' + 3y = x^2 - 2e^{2x}$

b) $y' \sin x + y \cos x = 1$

Exercice 2:

a) Trouver les valeurs a, b, c et d telles que :

$$\frac{1}{x^4 + 1} = \frac{ax + b}{x^2 + x\sqrt{2} + 1} + \frac{cx + d}{x^2 - x\sqrt{2} + 1}$$

b) Calculer $I = \displaystyle\int_0^1 \frac{1}{x^4 + 1} dx$

c) Déduire

i) $J = \displaystyle\int_0^1 \frac{1}{(x^4 + 1)^2} dx$

ii) $K = \displaystyle\int_0^1 \frac{x + 1}{(x^4 + 1)^2} dx$.

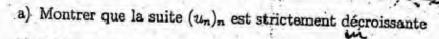**Exercice 3:** Calculer les limites suivantes en utilisant le DL

a) $\displaystyle\lim_{x \to 0} \frac{1}{x} - \frac{1}{\ln(1 + x)}$

b) $\displaystyle\lim_{x \to +\infty} x^2 [\ln(x + \sqrt{1 + x^2}) - \ln x]$
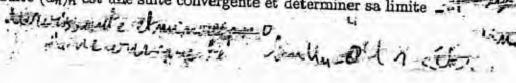
Exercice 4: Soit la suite récurente $(u_n)_n$ définie par :

$$\begin{cases} u_n := \ln(1 + u_{n-1}) \\ u_0 > 0 \end{cases}$$

a) Montrer que la suite $(u_n)_n$ est strictement décroissante

b) Déduire que la suite $(u_n)_n$ est une suite convergente et déterminer sa limite

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Exercice 1 a/ $(E) : 2y'' - 7y' + 3y = x^2 - 2e^{2x}$

. Resolution de $(E') : 2y'' - 7y' + 3y = 0$

l'eq caracteristique associée est : $2r^2 - 7r + 3 = 0$ $\Delta = 25$, $r_1 = 1/2$, $r_2 = 3$

$$y_1 = \alpha e^{x/2} + \beta e^{3x} \quad \text{sol de } (E')$$

. Solution particulière y_0 de (E) s'écrit $y_0 = y_{01} + y_{02}$ avec y_{01} sol particulière de

l'eq $(E_1) : 2y'' - 7y' + 3y = x^2$ et y_{02} sol particulière de $(E_2) : 2y'' - 7y' + 3y = -2e^{2x}$

. $y_{01} = ax^2 + bx + c$ car $\deg(x^2) = 2$ et $c \neq 0$

$y_{01}' = 2ax + b$, $y_{01}'' = 2a$

Alors $2y_{01}'' - 7y_{01}' + 3y_{01} = x^2 \Rightarrow 4a - 14ax - 7b + 3ax^2 + 3bx + 3c = x^2 \Rightarrow \begin{cases} 3a = 1 \\ -14a + b = 0 \\ 4a - 7b + 3c = 0 \end{cases}$

$\Rightarrow a = 1/3 ; b = \dfrac{14}{9} ; c = \dfrac{86}{27}$ $\Rightarrow y_{01} = 1/3 \, x^2 + \dfrac{14}{9} x + \dfrac{86}{27}$

. $y_{02} = a e^{2x}$ car $m = 2$ n'est pas racine de l'eq caractéristique

$y_{02}' = 2a e^{2x}$, $y_{02}'' = 4a e^{2x}$

Alors $2y_{02}'' - 7y_{02}' + 3y_{02} = -2e^{2x} ; 8a e^{2x} - 14a e^{2x} + 3a e^{2x} = -2e^{2x} \Rightarrow -3a = -2 \Rightarrow a = \dfrac{2}{3}$

$\Rightarrow y_{02} = 2/3 \, e^{2x}$ $\Rightarrow y_0 = y_{01} + y_{02} = 1/3 \, x^2 + \dfrac{14}{9} x + \dfrac{86}{27} + 2/3 \, e^{2x}$

et $y = y_1 + y_0 = \alpha e^{x/2} + \beta e^{3x} + 1/3 \, x^2 + \dfrac{14}{9} x + \dfrac{86}{27} + 2/3 \, e^{2x}$ sol de E

b/ $(E) : y' \sin x + y \cos x = 1$

. $(E') : y' \sin x + y \cos x = 0 \rightarrow \dfrac{y'}{y} = -\dfrac{\cos x}{\sin x} ; \ln|y| = -\ln|\sin x| + C \Rightarrow y_1 = \dfrac{K}{\sin x}$

. Determinons une solution particulière de E sous la forme $y_0 = \dfrac{K}{\sin x}$, avec K variable

$y_0' = \dfrac{K' \sin x - K \cos x}{\sin^2 x}$; Alors $y_0' \sin x + y_0 \cos x = 1 \Rightarrow \dfrac{K' \sin x - K \cos x}{\sin x} + \dfrac{K \cos x}{\sin x} = 1$

$\Rightarrow K' = 1 \Rightarrow K = x$ et $y_0 = \dfrac{x}{\sin x}$ d'où $y = y_1 + y_0 = \dfrac{K + x}{\sin x}$ sol de (E)

Exercice 2 a/ $\dfrac{1}{x^4 + 1} = \dfrac{ax + b}{x^2 + x\sqrt{2} + 1} + \dfrac{cx + d}{x^2 - x\sqrt{2} + 1} \Rightarrow 1 = (ax + b)(x^2 - x\sqrt{2} + 1) + (x^2 + x\sqrt{2} + 1)(cx + d)$

$(a + c) x^3 + (-a\sqrt{2} + b + c\sqrt{2} + d) x^2 + (a - b\sqrt{2} + c + d\sqrt{2}) x + b + d = 1 \Rightarrow \begin{cases} a + c = 0 \\ -a\sqrt{2} + b + c\sqrt{2} + d = 0 \\ a - b\sqrt{2} + c + d\sqrt{2} = 0 \\ b + d = 1 \end{cases}$

$\begin{cases} c = -a \\ d = 1 - b \\ -a\sqrt{2} + b - a\sqrt{2} - b + 1 = 0 \\ a - b\sqrt{2} - a - b\sqrt{2} + \sqrt{2} = 0 \end{cases} \Rightarrow \begin{cases} a = \sqrt{2}/4 \\ b = 1/2 \\ c = -\sqrt{2}/4 \\ d = 1/2 \end{cases}$

b/ $\dfrac{1}{x^4 + 1} = \dfrac{\frac{\sqrt{2}}{4} x + \frac{1}{2}}{x^2 + x\sqrt{2} + 1} + \dfrac{-\frac{\sqrt{2}}{4} x + \frac{1}{2}}{x^2 - x\sqrt{2} + 1} = \dfrac{\sqrt{2}}{4} \left(\dfrac{x + \sqrt{2}}{x^2 + x\sqrt{2} + 1} - \dfrac{x - \sqrt{2}}{x^2 - x\sqrt{2} + 1} \right)$

$= \dfrac{\sqrt{2}}{8} \left(\dfrac{2x + 2\sqrt{2}}{x^2 + x\sqrt{2} + 1} - \dfrac{2x - 2\sqrt{2}}{x^2 - x\sqrt{2} + 1} \right) = \dfrac{\sqrt{2}}{8} \left(\dfrac{2x + \sqrt{2} + \sqrt{2}}{x^2 + x\sqrt{2} + 1} - \dfrac{2x - \sqrt{2} - \sqrt{2}}{x^2 - x\sqrt{2} + 1} \right)$

$$\frac{1}{x^4+1} = \frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1} + \frac{\sqrt{2}}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} - \frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1} + \frac{\sqrt{2}}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)$$

$$F(x) = \frac{\sqrt{2}}{8}\left(\ln(x^2+x\sqrt{2}+1) + \sqrt{2}\cdot\sqrt{2}\, \text{Arctan}\left(\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right) - \ln(x^2-x\sqrt{2}+1) + \sqrt{2}\cdot\sqrt{2}\, \text{Arctan}\left(\frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)\right)$$

$$= \frac{\sqrt{2}}{8}\left(\ln\left(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right) + 2\,\text{Arctan}(\sqrt{2}x+1) + 2\,\text{Arctan}(\sqrt{2}x-1)\right)$$

$$\int_0^1 \frac{1}{x^4+1}\,dx = [F(x)]_0^1 = F(1)-F(0) = \frac{\sqrt{2}}{8}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\,\text{Arctan}(1+\sqrt{2}) + 2\,\text{Arctan}(\sqrt{2}-1)\right)$$

c/ i/ $\quad I = \int_0^1 \frac{1}{x^4+1}\,dx \qquad$ On pose $\begin{cases} u = \frac{1}{x^4+1} = (x^4+1)^{-1} \\ v' = 1 \end{cases} \begin{cases} u' = -4x^3(x^4+1)^{-2} \\ v = x \end{cases}$

$$I = \left[\frac{x}{x^4+1}\right]_0^1 - \int_0^1 \frac{-4x^4}{(x^4+1)^2}\,dx = \frac{1}{2} + 4\int_0^1 \frac{x^4}{(x^4+1)^2}\,dx = \frac{1}{2} + 4\int_0^1 \frac{x^4+1-1}{(x^4+1)^2}\,dx$$

$$I = \frac{1}{2} + 4\left(\int_0^1 \frac{x^4+1}{(x^4+1)^2}\,dx - \int_0^1 \frac{1}{(x^4+1)^2}\,dx\right) \Rightarrow I = \frac{1}{2} + 4I - 4J \Rightarrow 4J = 3I + \frac{1}{2}$$

$$J = \frac{3}{4}I + \frac{1}{8} = \frac{3\sqrt{2}}{32}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\,\text{Arctan}(1+\sqrt{2}) + 2\,\text{Arctan}(\sqrt{2}-1)\right) + \frac{1}{8}$$

ii/ $\quad K = \int_0^1 \frac{x+1}{(x^4+1)^2}\,dx = \int_0^1 \frac{x}{(x^4+1)^2}\,dx + \int_0^1 \frac{1}{(x^4+1)^2}\,dx = L + J$

$$L = \int_0^1 \frac{x}{(x^4+1)^2}\,dx = \int_0^1 \frac{\frac{1}{2}\,dt}{(t^2+1)^2} = \frac{1}{2}\int_0^1 \frac{dt}{(t^2+1)^2} \qquad \text{On pose : } x^2 = t \quad ; \quad 2x\,dx = dt$$

De plus $\quad \int_0^1 \frac{1}{t^2+1}\,dt = [\text{Arctan}\,t]_0^1 = \pi/4$

et $\quad \int_0^1 \frac{1}{t^2+1}\,dt = \left[\frac{t}{t^2+1}\right]_0^1 - \int_0^1 \frac{-2t^2}{(1+t^2)^2}\,dt \qquad \begin{cases} u = \frac{1}{t^2+1} = (t^2+1)^{-1} \\ v' = 1 \end{cases} \begin{cases} u' = -2t(t^2+1)^{-2} \\ v = t \end{cases}$

$$\frac{\pi}{4} = \frac{1}{2} + 2\int_0^1 \frac{t^2}{(1+t^2)^2}\,dt = \frac{1}{2} + 2\left(\int_0^1 \frac{t^2+1-1}{(1+t^2)^2}\,dt\right) = \frac{1}{2} + 2\left(\int_0^1 \frac{1}{1+t^2}\,dt - \int_0^1 \frac{1}{(1+t^2)^2}\,dt\right)$$

$$\frac{\pi}{4} = \frac{1}{2} + 2\left(\frac{\pi}{4} - 2L\right) \Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{\pi}{2} - 4L \Rightarrow L = \frac{\pi+2}{16}$$

d'où $\quad K = \frac{\pi+2}{16} + \frac{3\sqrt{2}}{32}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\,\text{Arctan}(1+\sqrt{2}) + 2\,\text{Arctan}(\sqrt{2}-1)\right) + \frac{1}{8}$

Exercice) a/ $\quad \frac{1}{x} - \frac{1}{\ln(1+x)} = \frac{\ln(1+x)-x}{x\ln(1+x)} = \frac{x-\frac{x^2}{2}+x^2\varepsilon(x)-x}{x(x-\frac{x^2}{2}+x^2\varepsilon(x))} = \frac{-\frac{1}{2}+\varepsilon(x)}{1+\frac{x}{2}\varepsilon(x)} \xrightarrow[x\to 0]{} -1/2$

$$\dots = x^2\left(\ln(x+\sqrt{1+x^2}) - \ln x\right) = x^2\left(\ln x + \ln\left(1+\frac{\sqrt{1+x^2}}{x}\right) - \ln x\right) = x^2\left(\ln x + \ln\left(1+\frac{\sqrt{1+x^2}}{x}\right) - \ln x\right)$$

$$= x^2 \ln\left(1+\frac{\sqrt{1+x^2}}{x}\right)$$

Posons $X = \frac{1}{x}$ avec $x>0$ alors $f(x) = \frac{1}{x^2}\ln\left(1+\frac{\sqrt{1+\frac{1}{x^2}}}{\frac{1}{x}}\right) = \frac{1}{x^2}\ln(1+\sqrt{1+x^2})$

$$\sqrt{1+x^2} = (1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 + x^2\varepsilon(x)$$

$$f(x) = \frac{1}{x^2}\ln\left(2+\frac{1}{2}x^2+x^2\varepsilon(x)\right) = \frac{1}{x^2}\left(\ln 2 + \ln\left(1+\frac{1}{4}x^2+x^2\varepsilon(x)\right)\right)$$

$$= \frac{1}{x^2}\left(\ln 2 + \frac{1}{4}x^2 + x^2\varepsilon(x)\right) = \frac{\ln 2}{x^2} + \frac{1}{4} + \varepsilon(x) = \frac{1}{4} + x^2\ln 2 + \varepsilon\left(\frac{1}{x}\right)$$

$$\lim_{+\infty} f(x) = +\infty$$

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